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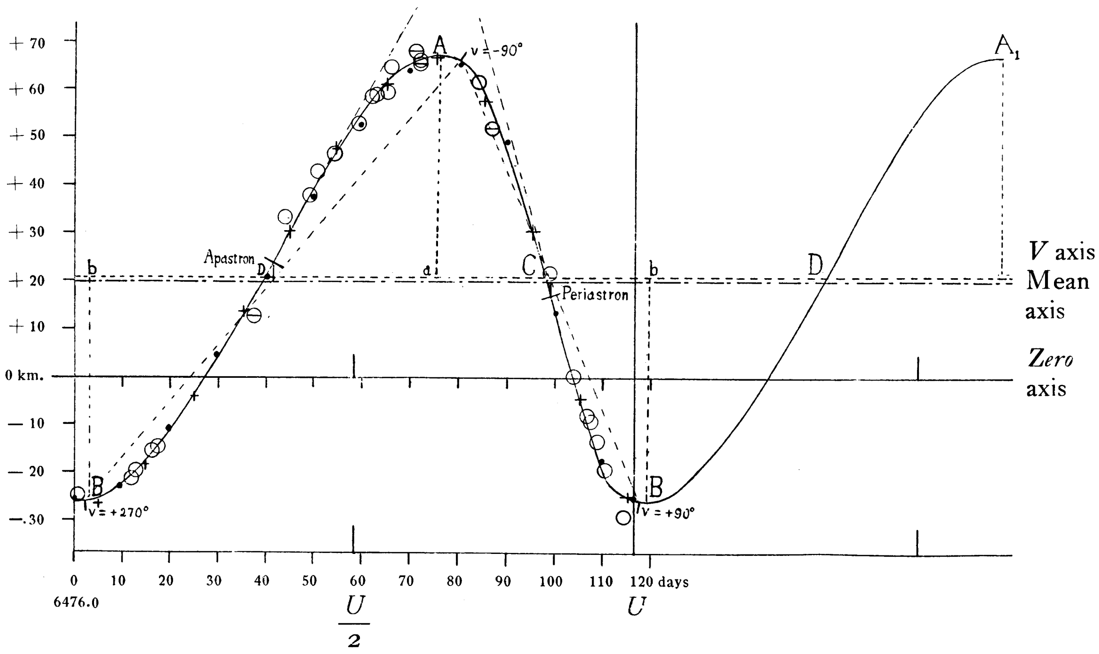+ 70
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v = -90°
v = +270°
v = +90°
Apastron
Periastron
V axis
Mean axis
Zero axis

FIGURE 2.

PUBLICATIONS

OF THE

Astronomical Society of the Pacific.

Vol. XX. San Francisco, California, June 10, 1908. No. 120.

METHODS OF DETERMINING THE ORBITS OF SPECTROSCOPIC BINARIES.

By Heber D. Curtis.

With the constantly increasing number of discoveries in this field of spectrographic research, the subject of spectroscopic binaries as a whole has become an exceedingly important one in its bearing on theories of stellar evolution. Thirty, or even twenty, years ago it would have seemed beyond the bounds of reasonable conjecture that at least one in every seven or eight of the brighter stars should prove to be a double star whose components are impossible of separation in any telescope existing or whose construction is conceivable. It is unnecessary to speak of the importance of the results already secured in relation to theories of cosmogony. As one conclusion only, we are led to think that our Sun, with its retinue of small planets, may possibly be a rather exceptional case in stellar development, the more common case being the fission of a star into two or more masses of the same order of magnitude.

The number of these systems is increasing to-day at a rapid rate. Including those whose binary character is under investigation, but not sufficiently substantiated for publication, the number is now about two hundred; and, with the carrying of spectrographic surveys down to fainter stars, we may need four figures to express the total in the not distant future.

It seems improbable at present that the labor of "keeping track" of all these systems will ever present the difficulties which make the asteroid group rather an embarrassing source of astronomical wealth, though we cannot, of course, premise just what interesting developments may be gained as to the inner mechanism of these systems from a comparison of their orbits of to-day with their forms one hundred years hence. It is not too much to suppose that the student of celestial mechanics may then find in this field "laboratory demonstrations" of the problem of three or more bodies, while possible changes in the periods and eccentricities could be expected to give valuable results to the worker in theories of tidal evolution of planets and satellites.

In the sphere of double stars we now have an unbroken continuum, from the spectroscopic binary of a few hours' rotation period to the visual double star with a period so long that centuries will be necessary for an accurate determination of the orbital elements. Some five or six stars are known to be both spectroscopically and telescopically double, and these serve to bind together the two fields. Visual double-star observers to-day are agreed that the greatest present progress in their branch of this wide field is to be derived from the systematic measurement of the systems of shorter period, from five to fifty years. It is in just this point of shortness of period that the spectroscopic field has the advantage in the point of evolutional developments. To take an extreme case: In the interesting quadruple system of *Castor* the published spectrographic observations of α_1, the fainter component, extend over eleven years and cover no less than fourteen hundred complete revolutions, while for the visual system the telescopic measures extending over one hundred years cover so short an arc that the visual elements must remain uncertain for a long time to come.

The actual determination of the elements of a spectroscopic binary is comparatively easy. From twenty to thirty plates, if well distributed over the period of revolution, are ample, in most cases, to secure satisfactory values of the elements of the system. The securing of these plates is the time-consuming factor in the problem, however, which perhaps explains why, for about seventy-five per cent of known spectroscopic binaries, we are still without accruate elements. While a skilful observer of visual double stars can make and reduce a complete single set of measures in fifteen to twenty minutes, including in this the time taken in passing from one star of his programme to another, from half an hour to two or more hours is required for the exposure time alone of a single spectrographic plate. Developing, measuring, and reducing, unless but few lines are available in the spectrum, will take several hours more. From a consideration of these points it may easily be seen that the determination of the orbits of three or four hundred spectroscopic binaries would require in the neighborhood of ten thousand plates,—easily twenty years' work for a single observatory, however favorably located as to climatic conditions.

Doubtless for some years to come the individual observatories possessing powerful spectrographs for radial-velocity determinations will endeavor to compute the elements of the binaries discovered by themselves. But it seems probable that the time must come when there will be systematic co-operation, to some extent at least, in order to secure greater rapidity in utilizing the data at present available and to be accumulated, and to prevent unnecessary duplication of work.[1]

As no recent treatment of the various methods for the determination of the elements of a spectroscopic binary exists in English, it is the purpose of this paper merely to collect in accessible form some of the more approved methods of solving the problem, with a few hints as to the processes involved, but without pretending to advance any new matter on the subject.

The more important papers on the subject may be referred to as follows:—

RAMBAUT, *M. N.*, Vol. LI, pp. 316 ff.
WILSING, *A. N.*, **134**, 90, 1893.
RUSSELL, *Aph. J.*, **15**, 252, 1902.
LEHMANN-FILHÉS, *A. N.*, **136**, 17, 1894.
SCHWARZSCHILD, *A. N.*, **152**, 65, 1900.
BAUSCHINGER, "Bahnbestimmung der Himmelskörper," pp. 647 ff.
ZURHELLEN, *A. N.*, **173**, 353, 1907.
ZURHELLEN, *A. N.*, **175**, 245, 1907.

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[1] For several years the Lick Observatory has been keeping records of work in this field,—a task necessitated by the need for systematizing its own efforts,—and as a result it was found wise to collect the material available in a "First Catalogue of Spectroscopic Binaries" (CAMPBELL and CURTIS, *Lick Observatory Bulletin*, No. 79, 1905). From time to time, as the need arises, successive catalogues will be issued.—W. W. C.

Of these methods, that of RAMBAUT is considerably longer and more involved than later methods and need not be considered here. WILSING'S method, as originally published, was suitable only for orbits of small eccentricity, but Dr. RUSSELL has extended it so as to make it applicable to larger eccentricities as well. The method of WILSING and RUSSELL is purely analytical, and in time consumed is considerably longer than the geometrical methods of LEHMANN-FILHÉS, SCHWARZSCHILD, and ZURHELLEN. It consists in finding a Fourier's series for the velocity in terms of the elements. It should be very useful in special cases, particularly when the period is so nearly a year that one part of the velocity-curve is not represented by any observations. Another special application suggested by RUSSELL is its use as a method of trigonometrical analysis in disentangling the superposed elements of a triple or other multiple system. These longer methods will not be further treated here.

The spectroscopic plates give us a series of irregularly distributed radial velocities of the star, and the first problem is the determination of the approximate period, which rarely offers any difficulty. This should be done as early as possible in order to distribute the plates evenly over the revolution period and to fill up any gaps which may have been left by earlier plates. Different series of plates will differ so widely in their arrangement that the determination of each particular period is a separate problem. The observation times are most conveniently expressed in Julian days and decimals of a day. By choosing an epoch, preferably a maximum or minimum near the beginning of the series, a probable approximate value of the period is easily secured by the use of co-ordinate paper. When a number of maxima and minima are well defined, wide divergencies in other regions of the curve may generally be removed by trying a new period, some sub-multiple of the first period assumed. Then by subtracting multiples of the adopted period from the observation dates all the observations are reduced to a curve covering one revolution beginning at the assumed epoch, with the abscissæ as times from the adopted date of maximum or minimum, and the observed velocities as the ordinates. After the preliminary curve is drawn, by noting the deviations from the curve at points near the mean of the maximum and minimum velocities, where a change in the periodic time will have the greatest effect, advisable changes in the preliminary assumed period are readily found by dividing the deviations of such critical observations, expressed in time, by the number of revolutions elapsed. A second curve may then be drawn, whose periodic time will generally be very close to the true one. For all this preliminary work a calculating machine is a great time-saver, though a simple table of multiples of the period will be almost as rapid. For stars whose periods are almost exact multiples of a day, and short, it may be necessary to wait some time before observations can be secured over all the velocity curve. The same may be true for orbits nearly a year in length, unless the analytical method of WILSING and RUSSELL be used in these cases.

After the period has been determined as carefully as possible the "most probable curve" is drawn, by estimation, through the observation points as plotted; and it is well to prolong this curve through a revolution and a half, in order to be able to use the ingenious methods of superposition devised by SCHWARZSCHILD and ZURHELLEN.

The errors in drawing this curve have considerable effect on the accuracy of the determination of the elements. The most probable curve which even a practiced hand may draw is not likely to be a perfect representation of the elliptic motion which caused it, as it is natural to bend the curve slightly in or out at various points to satisfy the more or less exact observations. This difficulty is fundamental in all methods of treatment, and for it there seems to be no remedy other than by testing rapidly the first orbit found by a trial ephemeris and making the needful small changes in the elements, to which reference will be made later.[1]

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[1] Though ZURHELLEN mentions this unavoidable curve-error earlier in his valuable paper, it appears that he has not duly taken it into account, in the comparisons which he makes at the close of his paper, of the values of ω and the eccentricity secured by his methods and those obtained by other investigators using other methods. His values, it is true, are quite close to the values secured, in some cases, by a rigorous least-square solution, but his values have evidently been derived from the *final* curves as published. These, of course, are accurately drawn from ephemerides and are very approximately true representations of the line of sight component of the velocity in an elliptic orbit. Had he applied his methods to the original preliminary curves used by the investigators with whose work he makes comparisons, he would doubtless have found values as far removed from the final ones as are the preliminary values secured by the method of LEHMANN-

Figure 2 gives a preliminary velocity-curve as actually drawn for the determination of the orbit of the spectroscopic binary κ *Velorum*. The observations used were as follows:—

Julian Day, G. M. T.	Vel.	Julian Day, G. M. T.	Vel.
2416546.739	+ 68.5^{km}	2417686.591	+ 33.8^{km}
60.703	+ 12.9	91.572	+ 38.2
97.651	+ 65.7	92.545	+ 43.2
6912.601	+ 53.3	96.480	+ 46.7
7587.844	+ 58.6	7701.494	+ 52.7
88.788	+ 57.9	41.466	+ 22.1
90.829	+ 58.5	46.463	+ 0.3
91.824	+ 64.8	49.470	— 7.6
97.788	+ 65.8	50.479	— 8.8
7609.790	+ 62.0	51.463	— 13.3
54.534	— 21.0	53.457	— 19.2
55.556	— 19.2	58.451	— 29.0
58.570	— 15.2	59.460	— 24.6
59.545	— 14.5		

The small circles representing the first four observations, which are important in determining the period, owing to their distance in time from the later ones, are barred in the diagram. The period, *U*, was assumed to be 116.65 days, and the beginning of the curve is at Julian day 2416476.0. This is not exactly at a minimum, as may be seen from the diagram.

The conditions of the problem are illustrated in Figure 1. The XY-plane is taken as the plane tangent to the celestial sphere at the center of motion, the Z-axis being the line of sight in which the velocities are measured, and perpendicular to the XY-plane. The orientation of the X and Y axes in space remains unknown. It will be seen at once that the

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FILHÈS. Taking the *final* curve of α_2 *Geminorum* as published, and using a planimeter, I find by the method of LEHMANN-FILHÈS the following values:—

Lehmann-Filhès.	Final Elements.
$e = 0.501$	$e = 0.503$
$\omega = 265^\circ.36$	$\omega = 265^\circ.35$

Using the same procedure for the curve of β *Aurigæ* as published with ZURHELLEN'S paper, I derive $e = 0.144$, in close agreement with the value from his method No. 8, ($e = 0.142$); for both his method No. 8 and that of LEHMANN-FILHÈS are equally sensitive to an erroneous placing of the maximum, which obtains in this curve. Shifting the maximum back a distance 0.01 U, as suggested in the paper quoted, I find $e = 0.11$ (ZURHELLEN, mean of eight methods, $e = 0.106$).

problem is somewhat different from that in which the orbit of a visual double is determined. In that case the data are the observed distances and position-angles at certain epochs, which, when plotted, give us a projected ellipse, while the spectrographic data are simply the times and the corresponding velocities directly toward or directly away from the observer (+ when the distance from the Sun is increasing). The true velocities or the direction of motion in the orbital ellipse, and the size and the position in space of the actual orbit, must of course always remain unknown, except in the comparatively rare cases where visual and spectrographic observations can be combined. It is manifest, moreover, that if the orbit-plane were coincident with the XY-plane, such a binary could not be detected spectroscopically, as there would then exist no component of the velocity in the line of sight.

The following customary notation will be used:—

U = Period,
μ = mean daily motion,
i = inclination of orbit to XY-plane (not determinable),[1]
Ω = ascending node on XY-plane (not determinable),
a = semi-major axis of true ellipse " "
p = semi-parameter of true ellipse " "
ω = angular distance of periastron from ascending node,
u = argument of the latitude, $u = v + \omega$,
r = radius vector,
t = the time,
T = the time of periastron passage,
v, M, E = true, mean, and eccentric anomalies respectively,
e = eccentricity, $= \sin\phi$, where ϕ is the eccentric angle,
V = the velocity of the center of mass of the system as a whole (ZURHELLEN'S γ).
f = constant of attraction.[2]

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[1] While a and i are separately indeterminable, it will be seen later that we can find the value of $a \sin i$, from which we may assign a minimum for a.

[2] In the great majority of spectroscopic binaries the spectrum of one component only is visible. This does not necessarily mean that the companion is dark, but merely that it is considerably fainter. In this case the motion of the bright star

By reference to Fig. 1 it will be seen that the distance from a point S in the orbit to the XY-plane will be

$$z = r \sin i \sin u.$$

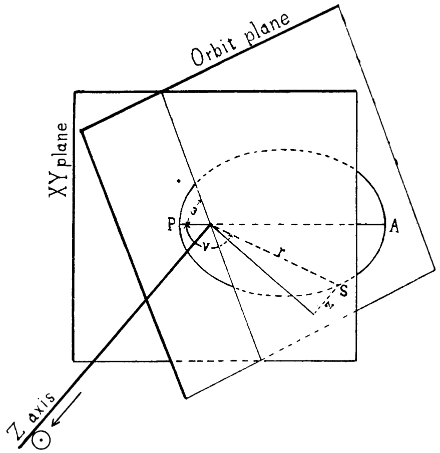

FIGURE 1.

The spectrograph gives us, however, not the distances from the XY-plane, but the *velocities* of the star's approach or recession from this plane, generally expressed in kilometers per second, and evidently equal to dz/dt.

Hence, differentiating,

$$\frac{dz}{dt} = \sin i \sin u \frac{dr}{dt} + r \sin i \cos u \frac{du}{dt}.$$

From the known laws of elliptic motion we may substitute the values

$$\frac{dr}{dt} = \frac{f}{\sqrt{p}} e \sin (u - \omega),$$

$$r \frac{du}{dt} = \frac{f\sqrt{p}}{r} = \frac{f}{\sqrt{p}} [1 + e \cos (u - \omega)].$$

These substitutions give for the fundamental equation

$$\frac{dz}{dt} = \frac{f}{\sqrt{p}} \sin i (\cos u + e \cos \omega). \qquad (1)$$

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is determined with reference to the center of gravity of the system and the constant of attraction has the value

$$\frac{k m^{\frac{3}{2}}}{m + m'},$$

where k is the Gaussian constant. If both components are visible in the spectrum and the motion of one mass with reference to the other is determined, $f = k\sqrt{m + m'}$.

Now the curve of the observed velocities will have the general appearance of a distorted sine-curve with a maximum and a minimum at the points A and B in Fig. 2, but it is evident that the curve as a whole is shifted a distance above or below the zero axis by an amount equal to the velocity of approach or recession of the binary system as a whole, and before the observed values can be made purely periodic the velocity of the center of mass of the system, which is a constant quantity for a single system, must be eliminated from the observation values. Methods of finding the position of the line representing the velocity of the center of mass of the system will be noted later, but for a study of some of the properties of the curve we will suppose for the present that this velocity of the system as a whole is known, and is represented by the dotted line in Fig. 2. This will be called the V-axis.

It is evident from equation (1) that for a maximum velocity $\cos u = 1$, and for a minimum $\cos u = -1$; i. e., at the point A of the curve, $u = 0$; and at B, $u = 180°$; or, A and B are the points corresponding to the star's passage through the ascending and descending nodes of its orbit, respectively. Taking, then, A and B as the magnitudes of the curve ordinates at the points of maximum and minimum reckoned from the V-axis, B being taken as a positive quantity, we have

$$\mathrm{A} = \frac{f}{\sqrt{p}} \sin\ i\,(1 + e\cos\omega),$$

$$\mathrm{B} = \frac{f}{\sqrt{p}} \sin\ i\,(1 - e\cos\omega).$$

From these equations it is clear that the term $\frac{f}{\sqrt{p}}\sin i$ is the half amplitude of the curve in Fig. 2. This will be denoted by K,—i. e.

$$\frac{f}{\sqrt{p}}\sin i = K = \frac{\mathrm{A}+\mathrm{B}}{2},$$

$$\frac{f}{\sqrt{p}}\sin i\ e\cos\omega = \frac{\mathrm{A}-\mathrm{B}}{2},$$

$$e\cos\omega = \frac{\mathrm{A}-\mathrm{B}}{\mathrm{A}+\mathrm{B}}.$$

The expression for the velocity measured from the zero-axis may then be written, by placing $V' = V + \frac{A - B}{2}$,

$$\frac{dz}{dt} = V' + K \cos u. \qquad (2)$$

Instead of using the ordinates A and B measured from the V-axis to find K, we may of course take the corresponding values from the zero-axis, replacing $\frac{A + B}{2}$ by $\frac{M_1 - M_2}{2}$, where M_1 and M_2 are the ordinates of maximum and minimum respectively, measured from the zero-axis.

Given the curve and the above fundamental equations, quite a variety of methods may be used to find the elements, all requiring about the same amount of time, providing that a planimeter is available when using the method of LEHMANN-FILHÉS. If the computer is not provided with one of these convenient instruments, the methods of SCHWARZSCHILD and ZURHELLEN will be more expeditious. Accordingly these methods will be treated first.

SCHWARZSCHILD'S METHOD OF FINDING THE PERIASTRON TIME, T.

Given M_1 and M_2, the observed velocities of maximum and minimum as measured from the zero axis, draw the line whose ordinate is $\frac{M_1 + M_2}{2}$. In Fig. 2, $M_1 = +66.4$, $M_2 = -26.8$, therefore $(M_1 + M_2)/2 = +19.8$. This line will be called the *mean* axis, and is represented in Fig. 2 by the dot-and-dash line. Mark on this axis the points $U/2$ and $3U/2$; then lay a piece of semi-transparent paper over the curve and copy on this the curve with the mean axis, marking also the points 0, $U/2$, U, and $3U/2$. Shifting this copy bodily along the mean axis for a distance $U/2$, rotate the copy 180° about the mean axis —i. e. turn the copy face downward on the original curve, keeping the mean axis in coincidence, and bringing the point 0 or U of the copy over the point $U/2$ of the curve. The curves will then cut one another in general in four points, of which two will be the points of periastron and apastron. The proper two points will be found without difficulty, for peri- and apastron must be separated in time by one half a revolution, and must, moreover, lie on different branches of the velocity-curve. To determine which is periastron we have the criteria: (1) at periastron the velocity-curve is steeper with regard to the axis than at apastron; (2) the curve is for a shorter time on that side of the mean axis on which the point of periastron lies. By following this procedure we find in Fig. 2 that periastron lies at the point + 98.8 days from the epoch, giving for the dates of periastron J. D. 2416458.15, 6574.80, etc.

This ingenious method is exceedingly good, though the writer prefers the method of LEHMANN-FILHÉS when the eccentricity is small. In these cases ω and T are quite indeterminate, and small errors in drawing the velocity-curve will be very troublesome.

METHOD OF DETERMINING THE ANGLE OF PERIASTRON, ω.

It may easily be verified that $V' = V + K\, e \cos \omega$ represents the position of the mean axis, and accordingly that the ordinate of any curve point, referred to the mean axis, may be secured from

$$\zeta = \frac{dz}{dt} - V' = K \cos u.$$

Now at periastron $v = 0$ and hence $u = \omega$; therefore, if we call the ordinate of periastron from the mean axis ζ_p, we have

$$\cos \omega = \frac{\zeta_p}{K}.$$

Or, better, we may use the ordinates of both peri- and apastron,

$$\cos \omega = \frac{\zeta_p - \zeta_a}{2K}.$$

Both of these methods are at their best when ω is near 90°. From Fig. 2 we take $\zeta_p = -3.2$; $K = 46.6$; hence $\omega = 93^\circ.9$. Or, $\zeta_p = -3.2$; $\zeta_a = +3.9$; $\zeta_p - \zeta_a = -7.1$; and $\omega = 94^\circ.4$. An entirely analogous equation may be used in case the position of the V-axis is known. This is generally most accurately determined by a planimeter and the method of LEHMANN-FILHÉS, to be given later, but may also be found from the mean axis by the requirement that the V-axis must lie at a distance equal to $-Ke\cos\omega$ above the mean axis. Calling the ordinates measured from the V-axis, z, we may find

$$\cos\omega = \frac{z_p - z_a}{2K}.$$

Another method of finding ω, due in its simplified form to ZURHELLEN, is best when ω is small. It consists in finding the points on the velocity-curve where the true anomaly is equal to ± 90°. From the fundamental equation,

$$\zeta_1 = \frac{dz_1}{dt} - V = K(\sin\omega + e\cos\omega), \text{ for } v = -90°,$$

and

$$\zeta_2 = \frac{dz_2}{dt} - V = K(-\sin\omega + e\cos\omega), \text{ for } v = +90°;$$

or,

$$\zeta_1 = +K\sin\omega,$$
$$\zeta_2 = -K\sin\omega.$$

Moreover, at the points $v = \pm 90°$,

$$E_1 = -E_2,$$
$$M_1 = -M_2,$$
$$t_1 - T = -(t_2 - T);$$

hence these points lie symmetrically in ζ to the mean axis, and in time to the periastron. They may accordingly be determined by rotating the curve copy 180° about the point of intersection of the ordinate of periastron with the mean axis, when the two intersections of the copy with the original curve will give the points sought. The point $v = +270°$ may be found in like manner. As a check on the location of these points draw the lines connecting the points $v = +270°$ and $v = -90°$, also that from $v = -90°$ to $v = +90°$. These lines should cut the mean axis at its intersections with the ordinates of apastron and periastron, respectively. Calling the ordinates measured from the mean axis ζ_1 for the point $v = -90°$, and ζ_2 for the point $v = +90°$, then

$$\sin\omega = \frac{\zeta_1 - \zeta_2}{2K}, \text{ or } \tan\omega = \frac{\zeta_1 - \zeta_2}{\zeta_p - \zeta_a}.$$

This method is at its least advantage in the example given in Fig. 2, for here ω is close to 90°. At $v = -90°$ in the figure we find its ordinate from the mean axis to be $+46.1$, while for $v=+90^\circ$ we find -45.4.

Hence $\sin\omega = 91.5/93.2$, and $\omega = 101^\circ.0$.

Other interesting properties of the velocity-curve and methods for finding the periastron-angle and eccentricity are derived by ZURHELLEN. For instance, by rotating the curve copy about the intersection of the periastron ordinate with the V-axis instead of with the mean axis, we may find the curve points where the eccentric anomalies are equal to $\pm 90^\circ$. But these additional methods will not in general offer any advantages over those already given.

METHODS OF DETERMINING THE ECCENTRICITY.

1. Perhaps the best is ZURHELLEN'S method of drawing the tangents to the curve at the points of periastron and apastron. These can be drawn with considerable accuracy, except when the periastron falls near a maximum or minimum of the curve, for in this case slight changes in the position of periastron will cause considerable changes in the inclinations of the tangent-lines. The expression for the tangent will be

$$\frac{d\zeta}{dt}=\frac{2\pi}{U}\cdot\frac{d\zeta}{dM}=\frac{2\pi}{U}\,\frac{1}{1-e\cos E}\cdot\frac{d\zeta}{dE}.$$

From $u = v + \omega$, and by introducing the known values

$$\cos v=\frac{\cos E-\sin\phi}{1-e\cos E},\qquad \sin v=\frac{\cos\phi\sin E}{1-e\cos E},$$

the original velocity equation will take the form

$$\zeta=\frac{dz}{dt}=K\cos\phi.\frac{\cos\phi\cos\omega\cos E-\sin\omega\sin E}{1-e\cos E};$$

whence

$$\frac{d\zeta}{dt}=\frac{2\pi}{U}K\cos\phi.\frac{-\cos\phi\cos\omega\sin E-\sin\omega\cos E+e\sin\omega}{(1-e\cos E)^3}.$$

At periastron $E=0$, and at apastron $E=\pi$; whence

$$\frac{d\zeta}{dt_p}=\frac{-2\pi K\cos\phi\sin\omega}{U(1-e)^2},$$

$$\frac{d\zeta}{dt_a}=\frac{+2\pi K\cos\phi\sin\omega}{U(1+e)^2}.$$

From these values

$$\frac{\tan_p}{\tan_a} = \left(\frac{1+e}{1-e}\right)^2 = -q^2,$$

$$\text{or} \quad e = +\frac{q-1}{q+1}$$

In Fig. 2 the value of the tangent at periastron is 104°.8, while at apastron it is 60°.9; whence $q = 1.45$ and $e = 0.18$.

2. Draw the V-axis and determine the ordinates from this axis of the maximum and the minimum, and also of peri- and apastron. Using z for these ordinates,

$$e = \frac{z_{\mathrm{max}} + z_{\mathrm{min}}}{z_{\mathrm{p}} \quad - z_{\mathrm{a}}}.$$

3. The following method is that of SCHWARZSCHILD as simplified by ZURHELLEN. It is best when ω is small, and to be applicable for eccentricities over 0.2 implies the use of SCHWARZSCHILD'S table, of which the necessary portion is given below. It depends upon the difference in time of the ordinates of the points $v = \mp 90°$, before determined.

Since $\tan \frac{1}{2} E = \tan \frac{1}{2} v \tan (45° - \frac{1}{2} \phi)$, for $v = \mp 90°$, we have $\tan \frac{1}{2} v = \mp 1$, and therefore $E_1 = -(90° - \phi)$ and $E_2 = +(90° - \phi)$. Similarly,

$$M_1 = -(90° - \phi) + \frac{\sin \phi \sin (90° - \phi)}{\sin 1''},$$

$$M_2 = +(90° - \phi) - \frac{\sin \phi \sin (90° - \phi)}{\sin 1''},$$

$$M_2 - M_1 = \frac{360°}{U}(t_2 - t_1) = 180° - 2\phi - \frac{\sin (180° - 2\phi)}{\sin 1''},$$

$t_2 - t_1$ may be read off directly from the diagram, and $90° - \phi$ can then be taken directly from SCHWARZSCHILD'S table. For small values of e ($\phi < 12°$, $e < 0.2$) the table may be replaced by the shorter formula,

$$\phi = 90° \left(\tfrac{1}{2} - \frac{t_2 - t_1}{U}\right).$$

From Fig. 2 we have $t_1 = 80.0$ days, $t_2 = 117.8$ days, hence $\frac{t_2 - t_1}{U} = 0.324$; whence from the table, $90° - \phi = 73°.7$ and $e = 0.28$. Using the shorter formula, $e = 0.27$. The divergence from the other values which have been secured is because ω is so near to 90°, the most unfavorable case for this method.

SCHWARZSCHILD'S TABLE FOR THE EQUATION

$$2\eta - \sin 2\eta = 2\pi\frac{t_2 - t_1}{U}.$$

η	$\frac{t_2 - t_1}{U}$	η	$\frac{t_2 - t_1}{U}$	η	$\frac{t_2 - t_1}{U}$
0°	0.0000	30°	0.0290	60°	0.1956
1	0.0000	31	0.0318	61	0.2040
2	0.0000	32	0.0348	62	0.2125
3	0.0000	33	0.0380	63	0.2213
4	0.0001	34	0.0414	64	0.2303
5	0.0001	35	0.0450	65	0.2393
6	0.0002	36	0.0488	66	0.2485
7	0.0004	37	0.0527	67	0.2578
8	0.0006	38	0.0568	68	0.2673
9	0.0008	39	0.0611	69	0.2769
10	0.0011	40	0.0656	70	0.2867
11	0.0015	41	0.0703	71	0.2966
12	0.0020	42	0.0751	72	0.3065
13	0.0025	43	0.0802	73	0.3166
14	0.0031	44	0.0855	74	0.3268
15	0.0038	45	0.0910	75	0.3371
16	0.0046	46	0.0967	76	0.3475
17	0.0055	47	0.1025	77	0.3581
18	0.0065	48	0.1085	78	0.3687
19	0.0077	49	0.1147	79	0.3793
20	0.0089	50	0.1212	80	0.3900
21	0.0103	51	0.1278	81	0.4008
22	0.0117	52	0.1346	82	0.4117
23	0.0133	53	0.1416	83	0.4226
24	0.0151	54	0.1488	84	0.4335
25	0.0170	55	0.1561	85	0.4446
26	0.0191	56	0.1636	86	0.4557
27	0.0213	57	0.1713	87	0.4667
28	0.0237	58	0.1792	88	0.4778
29	0.0262	59	0.1873	89	0.4889
30	0.0290	60	0.1956	90	0.5000

METHOD OF LEHMANN-FILHÉS.

It has not seemed advisable to segregate the separate steps in this method for inclusion with the other methods given before. The method of LEHMANN-FILHÉS will be found very generally useful, and little, if any, longer than other methods, providing a planimeter is used.

Given the observations and the velocity-curve, the first step is the fixing of the position of the velocity of the center of gravity of the system,—the so-called V-axis. This is found by the requirement that the integral of *dz/dt*,—that is, the area of the velocity-curve,—must be equal for the portions of the curve above the V-axis and below it. By far the easiest method of performing this integration is to use a planimeter. A line is first drawn by estimation as nearly in the correct position to fulfil the requirement of equal areas as can be judged by the eye. Placing the planimeter pointer at the intersection of this line and the curve, trace out the curve lying on one side of the line in, say, a clockwise direction, and, on reaching the starting point, traverse the portion of the curve on the other side of the line, in the counter-clockwise direction. The readings of the planimeter will then give directly the difference in the two areas, and the position of the V-axis may be shifted slightly to correct the difference. A second double circuit by the planimeter will generally be all that is necessary to secure an accurate value of the position of the V-axis, not depending simply upon the magnitude of maximum and minimum as in the previous methods, but supported by the entire curve. The increased accuracy is of little importance, however, as it will be found best to correct whatever value of V may be adopted by the residuals from the final ephemeris. If a planimeter is not available, doubtless the best method to equalize the areas above and below the curve will be to draw the curve on cross-section paper and make a count of the small squares, as suggested by RAMBAUT. The position may also be found by an approximate mechanical integration, as advised by LEHMANN-FILHÉS, but in such a case the shorter methods explained above for finding the elements would generally be given the preference.

Given the position of the V-axis, the ordinates should next be drawn at the points of maximum and minimum velocity

(Aa, Bb in Fig. 2). It is at this point that this method is weakest and most subject to error, for slight errors in the fixing of the positions of these ordinates may easily arise. It is best to check the positions of these ordinates by the requirement that the curve portion AaC must be equal in area to CbB, while DaA = BbD.

Considering V as eliminated from the velocities, at the curve points C and D, we have $dz/dt = 0$; hence, from the fundamental velocity equation,

$$\cos u = -e \cos \omega = -\frac{A - B}{A + B}.$$

The argument of the latitude, u_1, corresponding to the point C, lies on the way from the ascending to the descending node, hence sine u_1 is positive. Then,

$$\sin u_1 = \frac{2\sqrt{AB}}{A + B},$$

$$\cos u_1 = -\frac{A - B}{A + B} = -e \cos \omega;$$

while for u_2 at the point D

$$\sin u_2 = -\frac{2\sqrt{AB}}{A + B} = -\sin u_1,$$

$$\cos u_2 = -\frac{A - B}{A + B} = \cos u_1.$$

Denoting the area of AaC by Z_1, and bBD by Z_2 (it will be seen that these areas represent the distances of the star from the XY-plane when at the points of its orbit corresponding to the arguments of the latitude u_1 and u_2) and the radii vectors by r_1 and r_2, we have

$$Z_1 = r_1 \sin i \sin u_1,$$

$$Z_2 = r_2 \sin i \sin u_2 = -r_2 \sin i \sin u_1,$$

$$\frac{r_1}{r_2} = -\frac{Z_1}{Z_2}.$$

Then, since

$$r_1 = \frac{p}{1 + e \cos (u_1 - \omega)},$$

$$r_2 = \frac{p}{1 + e \cos (u_1 + \omega)},$$

$$\frac{1 + e \cos u_1 \cos \omega - e \sin u_1 \sin \omega}{1 + e \cos u_1 \cos \omega + e \sin u_1 \sin \omega} = - \frac{Z_1}{Z_2};$$

and, since $\cos u_1 = - e \cos \omega$,

$$\frac{\sin u_1 - e \sin \omega}{\sin u_1 + e \sin \omega} = - \frac{Z_1}{Z_2}.$$

Whence,

$$e \sin \omega = \frac{Z_2 + Z_1}{Z_2 - Z_1} \sin u_1,$$

which may be written in the form

$$e \sin \omega = \frac{2\sqrt{AB}}{A + B} \cdot \frac{Z_2 + Z_1}{Z_2 - Z_1}.$$

As it was found earlier from a consideration of the values A and B that

$$e \cos \omega = \frac{A - B}{A + B},$$

these two equations give all that is needed for a determination of e and ω. The values of A and B are taken from the curve, and the areas Z_1 and Z_2 are quickly integrated from the curve portions AaC and bBD by means of a planimeter (the latter area is considered negative in sign and the unit of area used is entirely immaterial as only the ratios of the areas are required).

At the time of passing through periastron $u = \omega$, whence from the fundamental equation

$$\frac{dz}{dt_p} = K (1 + e) \cos \omega.$$

Knowing K, e, and ω, determine this value, and the abscissa of the point on the curve which has the above value as ordinate will give the value of T when properly combined with the epoch used for the beginning of the curve. As at the points A, B, and A_1, u equals 0, π, and 2π, respectively, and as ω is known, there will be no ambiguity as to the position of $u = \omega$.

We may instead determine the eccentric anomaly at the point C where u_1 is known, and then find the time from periastron, t_1, corresponding to this value of u_1 by the well-known formulæ

$$\frac{\tan E_1}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{u_1 - \omega}{2},$$

$$\mu = \frac{2\pi}{U},$$

$$T = t_1 - \frac{E_1 - e \sin E_1}{\mu}.$$

The equation for the mean daily motion may be transformed as follows:—

$$\mu = \frac{2\pi}{U} = \frac{f}{a^{\frac{3}{2}}} = \frac{A+B}{2} \cdot \frac{\sqrt{p}}{\sin i} \cdot \frac{1}{a^{\frac{3}{2}}};$$

from which the following equation for the product of the separately indeterminable a and $\sin i$ may be derived. The factor 86,400 must be introduced because the unit of time for A and B is the second, while for μ it is the day.

$$a \sin i = 86{,}400 \frac{A+B}{2\mu} \sqrt{1-e^2} = [4.13833] \text{ K U} \sqrt{1-e^2}.$$

In the use of the method of LEHMANN-FILHÉS slight inaccuracies in drawing the preliminary curve when the eccentricity is small are less troublesome than in the methods of superposition given before. For eccentricities under 0.10 all methods are apt to give too large values of the eccentricity, and it is well in such cases to reduce the first value found considerably and test a small value first.

EXAMPLE OF METHOD OF LEHMANN-FILHÉS (FROM FIG. 2).

$A = 46.3$	$K = \frac{A+B}{2} = 46.6$
$B = 46.9$	$Z_1 = +0.168$
$A + B = 93.2$	$Z_2 = -0.259$
$A - B = -0.6$	$Z_2 + Z_1 = -0.091$
	$Z_2 - Z_1 = -0.427$

$\log \sqrt{AB}$	1.6684	$\log Z_2 + Z_1$	8.9590n
colog $(A + B)$	8.0306	colog $Z_2 - Z_1$	0.3696n
$\log (A - B)$	9.7782n	$\sin u_1$	0.0000
$\log 2$	0.3010	$e \sin \omega$	9.3286
$\sin u_1$	0.0000	$e \cos \omega$	7.8808n $(= - \cos u_1)$
$\cos u_1$	7.8088	$\tan \omega$	1.4478n
u_1	89°.63	ω	92°.04
		$\log e$	9.3289
		e	0.21

$\log (1 + e)$	0.0828	const. log	4.1383
$\cos \omega$	8.5519n	K	1.6684
$\log K$	1.6684	U	2.0669
	0.3031n	$\sqrt{1 - e^2}$	9.9902
ordinate $_p = - 2.0^{km}$		$a \sin i$	7.8638
$t_p = 98.4$ days			$= 73{,}000{,}000^{km}$

$T =$ J. D. 2416457.75

(The longer method gives 6458.0.)

IMPROVED ELEMENTS.

Whatever the method used to determine the elements, it will always be necessary to test the elements as derived and make small changes in them before they can be regarded as a satisfactory representation of the observations. To do this a test ephemeris is rapidly computed for ten or twelve points of the curve. V being, as before, the velocity of the system as a whole, and placing $V' = V + K e \cos \omega = V + \frac{A - B}{2}$, the necessary formulæ for the ephemeris are

$$\frac{dz}{dt} = V' + K \cos u,$$

$$\tan \tfrac{1}{2} (u - \omega) = \sqrt{\frac{1 + e}{1 - e}} \tan \tfrac{1}{2} E,$$

$$M = E - e \sin E,$$

$$M = \mu (t - T).$$

Collecting the elements as derived by the method of LEHMANN-FILHÉS, with the necessary constants, we have:—

$U = 116.65$ days $\qquad \mu^\circ = 3^\circ.0861$

$e = 0.21 \qquad \log\sqrt{\frac{1+e}{1-e}} = 0.0926$

$T = 2416458.0 \qquad \frac{A-B}{2} = -0.3$

$K = 46.6 \qquad V' = +20.4$

$\omega = 92^\circ.04$

$V = +20.7^{\text{km}}$

Then, at the point where $t = 20$ days, on Fig. 2, we have the following sample from the test ephemeris:—

$t = 2416496.0$

$t - T = 38.0 \qquad \frac{1}{2}(u - \omega) = 79^\circ.23$

$M = 148^\circ.13 \qquad u - \omega = 158.46$

$E = 153.50$ (From Åstrand's Hülfstafeln) $\qquad u = 250.50$

$\tan \frac{1}{2} E = 0.6281 \qquad$ nat $\cos u = -0.334$

$\tan \frac{1}{2}(u - \omega) = 0.7207 \qquad V' + K \cos u = dz/dt = +4.8^{\text{km}}$

In this way a test ephemeris was rapidly computed for the points 0, 10, 20 days, etc., of Fig. 2. The resulting curve, to save confusion, is not drawn in, but its position is represented by the heavy black dots. A study of the discrepancies shows that e is a little too large, the branch on the apastron side being somewhat too sloping, while the other slope is a little too steep. When e is changed we shall also need to make small changes in T and ω. Making one or two changes in this way, and testing at critical points of the curve, the following elements were finally decided upon as best satisfying the observations:—

$V = +21.9^{\text{km}}$

$U = 116.65$ days

$e = 0.19$

$K = 46.5$

$\omega = 96^\circ.23$

$T =$ J. D. 2416459.0

$a \sin i = 73{,}000{,}000^{\text{km}}$

The correction to the value of V was found last of all from the residuals of the final ephemeris by the simple formula

$\frac{[v]}{n}$, where n is the number of observations and v the residual, o — c. The residuals from the final ephemeris and the final curve may be found in *Lick Observatory Bulletin,* No. 122, 1907.

Combining the best results from the methods of Schwarzschild and Zurhellen, we would have the following elements:—

$$T = 2416458.15$$
$$\omega = 94.4$$
$$e = 0.18$$
$$K = 46.6$$
$$V' = +\ 19.8$$
$$\log\sqrt{\frac{1+e}{1-e}} = 0.0790$$

A test curve from these elements was also computed, and its position is represented by the small crosses at the points 5, 15, 25, etc., days. These elements are seen also to need small corrections, but on the whole are rather nearer the observation curve than are the values secured by the method of Lehmann-Filhés, as the position of periastron is about the most favorable for the determination of the eccentricity by Zurhellen's methods of tangents.

With some experience it is possible to secure elements which would be little, if any, bettered by a least-square solution, unless a large number of well-determined velocities are available, extending over a considerable interval of time. For completeness, however, the least-squares formulæ are given here as derived by Lehmann-Filhés:—

$$\delta\,\frac{dz}{dt} = (\cos u + e\cos\omega)\ \delta K$$
$$+\,K\ \left[\cos\omega - \frac{\sin u\sin v}{1-e^2}\,(2 + e\cos v)\right]\ \delta e$$
$$-\,K\ (\sin u + e\sin\omega)\ \delta\,\omega$$
$$-\,K\sin u\ (1 + e\cos v)^2.\ (t - T)\ \frac{1}{(1-e^2)^{\frac{3}{2}}}\ \delta\,\mu$$
$$+\,\sin u\ (1 + e\cos v)^2\ \frac{K\mu}{(1-e^2)^{\frac{3}{2}}}\ \delta T.$$

In case both spectra are visible and the motion of one star is determined with reference to the other $f = k\sqrt{m + m'}$, and an equation for the relation of the masses is found from the relation

$$\mu = \frac{k\sqrt{m + m'}}{a^{\frac{3}{2}}},$$

giving

$$m + m' = \frac{\mu^2 (a \sin i)^3}{k^2 \sin^3 i},$$

where A, B, and $a \sin i$ are to be expressed in terms of the mean distance of the earth from the Sun. If the velocity of the center of mass of such a system has also been determined, then from this the relation of both masses is known, for each mass may then be represented separately in the form

$$\frac{\text{constant}}{\sin^3 i}.$$

Further data as to methods of treating the comparatively rare cases where both spectra are measurable are given in ZURHELLEN'S last paper.

In the still rarer cases where both visual and spectrographic observations of a system are at hand, valuable results as to the parallax and relative masses of such systems may be secured. Reference may be made here to Professor HUSSEY'S determination of the parallax of δ *Equulei* (*Lick Observatory Bulletin,* No. 32), Professor WRIGHT'S value of the parallax of α *Centauri* (*Lick Observatory Bulletin,* No. 60), and to Dr. CAMPBELL'S discussion of the variable radial velocity of *Sirius* (*Lick Observatory Bulletin,* No. 70).

THE D. O. MILLS EXPEDITION,
SANTIAGO, CHILE, October, 1907.